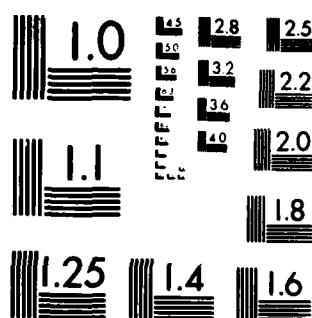


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A VERY SMALL REMARK ON  
SMOOTH DENJOY MAPS OF THE CIRCLE

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ABSTRACT

In this note we show that if  $f$  is a smooth homeomorphism of the circle with one 'non degenerate' critical point such that  $f$  has an irrational rotation number which is poorly approximable by rationals (e.g., a quadratic irrational) and  $f$  is a Denjoy map, then the length of the associated wandering interval cannot tend to zero too rapidly.

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### SIGNIFICANCE AND EXPLANATION

Maps of a circle into itself, when considered as discrete dynamical systems, have been observed to contain some of the qualitative behavior of complicated experimental situations. Much of this structure can be understood for homeomorphisms of the circle by considering the invariant called the rotation number. In fact, the homeomorphisms of the circle with irrational rotation number can be classified up to topological conjugacy, provided the map and its inverse are sufficiently smooth. Counter examples to this classification scheme exist which are  $C^1$  diffeomorphisms or  $C^\infty$  homeomorphisms with non differentiable inverse. Whether  $C^\infty$  can be replaced with analytic in the second examples is an open question.

In this report <sup>the authors</sup> shows that if certain assumptions are made on the rotation number and on the derivative of a homeomorphism from the circle onto itself, then even if the map does not fit into the classification scheme we can still obtain estimates on the behavior of its orbits under iteration.

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A VERY SMALL REMARK ON  
SMOOTH DENJOY MAPS OF THE CIRCLE

Glen Richard Hall

1. Introduction: Let  $T = \mathbb{R}/\mathbb{Z}$  be the circle with unit circumference. We say a homeomorphism  $f : T \rightarrow T$  is a Denjoy map if  $f$  has no periodic and no dense orbits. A theorem of Denjoy [2] states that if  $f$  is a  $C^2$  diffeomorphism then  $f$  is not a Denjoy map. He also constructs  $C^1$  diffeomorphisms which are Denjoy maps (see Herman [4]). Recently, examples of homeomorphisms  $f : T \rightarrow T$  which are  $C^\infty$  Denjoy maps have been constructed ([3]). These maps fail to be diffeomorphisms because they have points where the derivative of  $f$  is zero. However, the question of the existence of analytic homeomorphisms which are Denjoy maps remains open.

It turns out that a homeomorphism  $f : T \rightarrow T$  is a Denjoy map if and only if it has no periodic orbits and there exists a "wandering" interval  $I \subseteq T$  such that  $\{f^n(I)\}_{n=-\infty}^\infty$  forms a disjoint family of intervals. An interesting property of the examples of [3] is that they can be constructed so that length  $f^n(I) \rightarrow 0$  arbitrarily rapidly. (This requires an addition to [3] which states that "case B" will not occur. This was shown to the author by J. C. Yoccoz (personal communication).)

In this note we consider homeomorphisms  $f : T \rightarrow T$  with only a "non-degenerate" critical point (e.g., analytic homeomorphisms) and rotation number which is very poorly approximable by rationals (e.g., quadratic irrationals). If  $f$  is a Denjoy map with  $I \subseteq T$  a wandering interval as described above, then we show length  $f^n(I)$  cannot tend to zero arbitrarily rapidly.

As we will note in section 3, recent computer studies (Shenker, [5]) indicate that the estimates of this note might be significantly improved, yielding a corresponding improvement of the theorem.

2. Definitions and Notations: Fix a map  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following conditions

- 1)  $f_0$  is increasing,  $C^3$  and  $f_0(0) \in [0,1]$ ,
- 2)  $\forall x \in \mathbb{R}$ ,  $f_0(x+1) = f_0(x) + 1$ , (i.e.  $f_0$  is the lift of a homeomorphism of the circle),
- 3) there exists precisely one point  $x_0 \in [0,1]$  such that  $f_0'(x_0) = 0$  (hence  $f_0''(x_0) \neq 0$ ),
- 4)  $f_0'''(x_0) \neq 0$ ,
- 5)  $f_0''(x) = 0$  for only finitely many  $x \in [0,1]$ .

We let  $f(\cdot, \phi) : \mathbb{R} \rightarrow \mathbb{R}$  denote the one parameter family of maps given by

$$f(x, \phi) = f_0(x) + \phi .$$

Notation: Fix an interval  $J \subseteq [0,1]$ ,  $x_0 \in \text{interior } J$ . Then let

$$M_1 = \sup_{x \in [0,1]} \left| \frac{\partial f}{\partial x}(x, \phi) \right| ,$$

$$M_2 = \sup_{x \in [0,1]} \left| \frac{\partial^2 f}{\partial x^2}(x, \phi) \right| ,$$

$$M_3 = \inf_{x \in J} \left| \frac{\partial^3 f}{\partial x^3}(x, \phi) \right| ,$$

$$M_4 = \inf_{x \in [0,1] \setminus J} \left| \frac{\partial f}{\partial x}(x, \phi) \right| .$$

We may choose  $J$  small so that  $M_3 > 0$ . (Clearly these constants are independent of  $\phi$ .) We define  $f^n(x, \phi) = f(f^{n-1}(x, \phi), \phi)$  for  $n \in \mathbb{Z}$ , i.e.,  $f^n(x, \phi)$  is the  $n^{\text{th}}$  iterate of  $f(\cdot, \phi)$  with  $\phi$  fixed.

Definition: The rotation number of  $f(\cdot, \phi)$  denoted  $\rho(\phi)$  is given by

$$\rho(\phi) = \lim_{n \rightarrow \infty} \frac{f^n(x, \phi)}{n} .$$

Remarks: 1) The rotation number  $\rho(\phi)$  exists and is independent of the  $x$  in the definition. Moreover,  $\rho(\phi)$  is a continuous increasing function of  $\phi$ .

2) The function  $\rho(\phi)$  is strictly increasing when its value is irrational, i.e., if  $\rho(\bar{\phi}) \notin \mathbb{Q}$  and  $\phi \neq \bar{\phi}$  then  $\rho(\phi) \neq \rho(\bar{\phi})$  (see Herman [4] for proofs of these remarks).

Notation: For each  $\beta \in \mathbb{R}$ , let  $A_\beta = \{\phi : \rho(\phi) = \beta\}$ . Then for every  $\beta$ ,  $A_\beta$  is a closed interval, which is a single point when  $\beta$  is irrational.

Finally, we fix  $a_0 \in [0, 1]$ ,  $a_0$  an irrational such that there exists a constant  $M_5 > 0$  so that if

$$a_0 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \quad a_i \in \mathbb{Z}.$$

Then  $|a_i| < M_5$  for all  $i$ , (e.g., any quadratic irrational will satisfy this condition). We let  $\frac{p_n}{q_n}$  denote the  $n^{\text{th}}$  convergent of  $a_0$ , i.e.

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}.$$

Let  $\phi_0 \in \mathbb{R}$  be the unique number such that

$$\rho(\phi_0) = a_0.$$

3. Statement of the theorem: Suppose  $f(\cdot, \phi_0)$  is a Denjoy map, i.e.  $f(\cdot, \phi_0)$  is not continuously conjugate to rigid rotation by its rotation number  $a_0$ . Then there exists a nontrivial (i.e., non empty, non singleton)

interval  $I \subseteq [0,1]$  such that  $\{f^n(x) + m : x \in I\}_{n, \text{mez}}$  form a family of disjoint intervals. Let  $\mu_n = \text{length}(f^n(I))$ , then  $\mu_n \rightarrow 0$  as  $n \rightarrow \infty$  (recall  $f$  is the lift of a circle map).

Theorem: There exists a constant  $M_0 > 1$  such that

$$(3.1) \quad \mu_{\frac{p_n}{q_n}} > \frac{1}{M_0} \mu_n^{q_n} .$$

infinitely often.

That is, the length of the wandering interval cannot decrease too fast.

The outline of the proof is as follows: First we derive an estimate on the length of the intervals  $\mu_{\frac{p_n}{q_n}}$ . Next we show that the orbit of the critical point must return to a neighborhood of the critical point in each set of  $q_n$  iterates, moreover, the size of this neighborhood is controlled by the size of  $\mu_{\frac{p_n}{q_n}}$ . Hence, if  $\mu_{\frac{p_n}{q_n}}$  is very small, a small change in  $\phi$  will make a relatively large change in the rotation number, not leaving "enough room" for the  $\mu_{\frac{p_n}{q_n}}$  intervals.

Remark: It is important to note that the estimates we use on the derivatives of high iterates of  $f(\cdot, \phi)$  are very crude (see lemma 1 below). Recent numerical studies (Shenker, [5]) indicate much better estimates may hold, and if so, much better bounds on  $\mu_{\frac{p_n}{q_n}}$  would result.

4. The lemmas: In this section we give the necessary lemmas.

Lemma 1: For all  $x \in \mathbb{R}$ ,  $\phi \in \mathbb{R}$  and  $n \in \mathbb{Z}^+$

$$(4.1) \quad \left| \frac{\partial (f^n)}{\partial x} (x, \phi) \right| \leq M_1^n ,$$

$$(4.2) \quad \left| \frac{\partial^2 (f^n)}{\partial x^2} (x, \phi) \right| \leq M_2 \cdot M_1^{2n-2} / (M_1 - 1) ,$$

$$(4.3) \quad \left| \frac{\partial (f^n)}{\partial \phi} (x, \phi) \right| \leq M_1^{n+1} / (M_1 - 1) .$$

Proof: These estimates follow immediately from the chain rule and (2.1),

(2.2) (see remark, end of section 3). //

Lemma 2: For any  $n > 0$ ,

$$\text{length}(A_{p_n/q_n}) > 1/\left(\frac{4M_2 \cdot M_1}{(M_1-1)}\right)^{3q_n}.$$

Proof: Fix  $n > 0$  and let  $[\phi_1, \phi_2] = A_{p_n/q_n}$ . Then

$$\forall x \in \mathbb{R}, f^{q_n}(x, \phi_1) < x + p_n,$$

and there exists  $x_1 \in [0, 1]$  with

$$f^{q_n}(x_1, \phi_1) = x_1 + p_n.$$

Now,  $\frac{\partial f^{q_n}}{\partial x}(x_0, \phi) = 0$ , so by (4.2) for  $\delta = 1/\left(\frac{2M_2 \cdot M_1}{(M_1-1)}\right)^{2q_n-1}$  we have for any  $\phi$ ,

$$\frac{\partial f^{q_n}}{\partial x}(x, \phi) < \frac{1}{2} \text{ whenever } |x_0 - x| < \delta.$$

But then it follows that for any  $\phi$ ,

$$\begin{aligned} \max_{x \in [0, 1]} (x - f^{q_n}(x, \phi)) - \min_{x \in [0, 1]} (x - f^{q_n}(x, \phi)) \\ > 1/\left(\frac{4M_2 \cdot M_1}{(M_1-1)}\right)^{2q_n-1}. \end{aligned}$$

By (4.3) we then have that there exist  $x_2, x_3 \in \mathbb{R}$  such that

$$f^{q_n}(x_2, \phi) < x_2 + p_n \text{ and } f^{q_n}(x_3, \phi) > x_3 + p_n$$

whenever  $0 < \phi - \phi_1 < \frac{1}{(q_n+1)^{2q_n-1}} \left(\frac{4M_2 \cdot M_1}{(M_1-1)^2}\right)$  and the proof is complete. //

The following lemmas are standard results on the orbit structure for maps with irrational rotation number.

Lemma 3: The sets  $\{f^i(x, \phi_0) + m : i, m \in \mathbb{Z}, 0 < i < q_n\}$  and  $\{f^{-1}(x, \phi_0) + m : i, m \in \mathbb{Z}, 1 < i < q_n\}$  alternate on  $\mathbb{R}$  (i.e. between any two points of the first set there is a point of the second set and vice versa) for any fixed  $x \in \mathbb{R}$  and  $q_n$ .

Lemma 4: For each  $n > 0$ ,  $x_1, x_2 \in \mathbb{R}$ , if for some  $q_n > i, j > 0, m_1, m_2 \in \mathbb{Z}$ ,  $f^i(x_1, \phi_0) + m_1 < f^j(x_2, \phi_0) + m_2$  then there exists  $k$  and  $m_3$  with  $0 < k < q_{n+1}$  with

$$f^i(x_1, \phi_0) + m_1 < f^k(x_2, \phi_0) + m_3 < f^j(x_1, \phi_0) + m_2.$$

The proofs of these lemmas follow from, for example, Theorem 4.1, page 409 of Coddington, Levinson [1] and II. 7.1 of Herman [4], respectively.

Lemma 5: For  $n$  sufficiently large, the set  $\{f^i(y, \phi_0) - [f^i(y, \phi_0)] : i = 0, \dots, n\}$  (where  $[\cdot]$  denotes greatest integer) contains more than two points in  $J$  for any  $y \in [0, 1]$ .

Proof: It suffices to show that the orbit of any point, say  $x_0$ , contains the critical point in its closure. But if  $\{f^i(x_0, \phi_0) - [f^i(x_0, \phi_0)] : i > 0\}$  does not contain  $x_0$  then by altering  $f$  in a neighborhood of  $x_0$  we can make a  $C^3$  diffeomorphism without altering the orbit of  $x_0$ . (see Figure 1) which contradicts Denjoy's theorem and the proof is complete. //

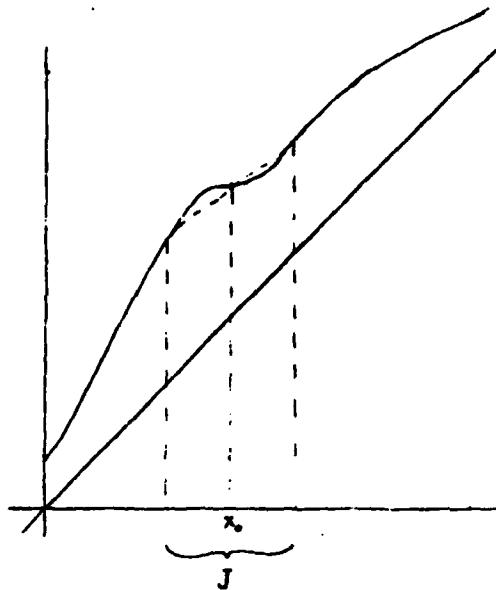


Figure 1

Remark: In fact the orbit of  $x_0$  must approach  $x_0$  from "both sides" as was shown by J. C. Yoccoz (personal communication).

5. Proof of the theorem. Without loss of generality we may assume  $\mu_0 = \max \mu_i$ . Now fix  $n > 0$  and for  $y, \hat{y} \in I$  let  $y_i = f^i(y, \phi_0) - [f^i(y, \phi_0)]$ ,  $i \in \mathbb{Z}$ ,  $\hat{y}_i = f^i(\hat{y}, \phi_0) - [f^i(\hat{y}, \phi_0)]$  where  $[\cdot]$  denotes the greatest integer. Fix  $i_0$  with  $0 < i_0 < q_n$  and  $|y_{i_0} - x_0| = \min\{|y_i - x_0| : i = 0, \dots, q_n - 1\}$ . By condition (5) in the definition of  $f_0$  there exist, say  $r$ , points in  $[0, 1)$  such that  $\frac{\partial^2 f}{\partial x^2}(x, \phi_0) = 0$ . Hence by lemma 3 we see that taking  $n$  sufficiently large, so that the conclusion of lemma 5 holds,

$$\begin{aligned} \frac{\partial (f_n)}{\partial x}(y, \phi_0) &= \prod_{j=0}^{q_n-1} \frac{\partial f}{\partial x}(y_j, \phi_0) \\ &> \prod_{j=1}^{q_n} \frac{\partial f}{\partial x}(\hat{y}_{-j}, \phi_0) \cdot \frac{r-1}{4} \cdot \frac{\partial f}{\partial x}(y_{i_0}, \phi_0) . \end{aligned}$$

This follows from lemma (3) since for each  $i$ ,  $0 < i < q_n$  there corresponds  $j$ ,  $0 > j > -q_n$  such that  $\frac{\partial f}{\partial x}(y_j, \phi_0) < \frac{\partial f}{\partial x}(y_i, \phi_0)$  except for  $i = i_0$  and the  $r-1$  choices of  $i \neq i_0$  where  $y_{i_0}$  is adjacent to a point with  $\frac{\partial^2 f}{\partial x^2} = 0$ . Hence, by an appropriate choice of  $\hat{y} \in I$  we obtain

$$\begin{aligned} \frac{\partial f}{\partial x}(y, \phi_0) &> \frac{\mu_0}{\mu_{-q_n}} M_4^{r-1} \frac{\partial f}{\partial x}(y_{i_0}, \phi_1) \\ &> M_4^{r-1} \frac{\partial f}{\partial x}(y_{i_0}, \phi_1) . \end{aligned}$$

Noting that

$$\frac{\partial f}{\partial x}(y_{i_0}, \phi_1) > 6M_3 \cdot (y_{i_0} - x_0)^2$$

we see that

$$\frac{\partial f}{\partial x}(y, \phi_0) > 6M_4^{r-1} \cdot M_3 (y_{i_0} - x_0)^2$$

for all  $y \in I$ . Hence, for some  $y \in I$

$$\begin{aligned} \mu_{q_n} &> 6\mu_0 M_4^{r-1} \cdot M_3 (y_{i_0} - x_0)^2, \text{ i.e.,} \\ (5.1) \quad (\mu_{q_n} / (\mu_0 \cdot 6M_4^{r-1} \cdot M_3))^{1/2} &> |y_{i_0} - x_0| . \end{aligned}$$

Suppose  $\mu_{q_n} < 1/c_n^{q_n}$  for all  $n$ , for some  $c > 1$ . Then, by (5.1), for each  $n$  there exists  $y_n \in I$  such that if  $\delta_n = \min_{i=0, \dots, q_n-1} |f^i(y_n, \phi_0) - [f^i(y_n, \phi_0)] - x_0|$  then

$$\delta_n < c_2/c_1^{q_n}$$

where  $c_1 = c^{1/2}$  and  $c_2 = (6\mu_0 M_4^{r-1} M_3)^{-1/2}$ . Fix  $m$  such that  $\delta_{m+1} < \delta_m$ .

Then by lemma 4 there exists  $t$ ,  $0 < t < q_{m+2}$  with

$$|f^t(x_0, \phi_0) - [f^t(x_0, \phi_0)] - x_0| < \delta_m .$$

But since  $1 < \frac{\partial f^n}{\partial \phi}(x, \phi)$  for all  $x, \phi, n$  we see that there exists  $\epsilon$  such that  $f^t(x_0, \phi_0 + \epsilon) - [f^t(x_0, \phi_0 + \epsilon)] = x_0$  and  $|\epsilon| < \delta_m$ . But then either  $A_{p_{m+3}/q_{m+3}} \subseteq [\phi_0 - |\epsilon|, \phi_0 + |\epsilon|]$  or  $A_{p_{m+4}/q_{m+4}} \subseteq [\phi_0 - |\epsilon|, \phi_0 + |\epsilon|]$  and hence by lemma 2

$$\delta_m > |\epsilon| > (M_1 - 1)^2 / (4M_2 \cdot M_1^{3q_{m+4}}) > (M_1 - 1)^2 / (4M_2 \cdot M_1^{3(M_4+1)^4 q_m})$$

i.e.  $c_2/c_1^{q_m} > (M_1 - 1)^2 / (4M_2 \cdot M_1^{3(M_4+1)^4 q_m})$ . Hence, we must have  $c_1 < M_1$ ,

which completes the proof of (3.1) for  $\mu_{q_n}$ . To show  $\mu_{-q_n}$  also satisfies

the theorem for  $M_0 = M_1^{9(M_4+1)^8} + 1$  is similar. //

6. Example: Let  $f_0(x) = x + \frac{1}{2\pi} \sin(2\pi x)$ . Then the family  $f(x, \phi) =$

$x + \phi + \frac{1}{2\pi} \sin(2\pi x)$  satisfies the hypotheses, and if we let  $a_0 = \frac{\sqrt{5} - 1}{2}$

then we obtain  $M_0 = 2^{2304} + 1$ .

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